A Unique Step Function and Stitching Piecewise Defined Functions

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May 26, 2021

Abstract

This paper discusses a unique complex logarithmic function that can be used as an alternative to piecewise defined step functions such as the Heaviside function. It describes how to create second tier logical functions such as multi-steps, boxcars, and valleys, and it provides a method to stitch the components of piecewise functions into a single function.

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1 Introduction

In many situations it is useful to define piecewise functions, with one of the most common examples being the Heaviside step function. This often leads to special considerations and additional actions that must be made when handling each part of the function, especially for procedures such as taking the derivative or integral. As such, it may sometimes be convenient to have a step function that is singular rather than piecewise, and it may also be useful to turn other more general piecewise defined functions into singular forms.

As described in the abstract, this paper discusses a unique complex logarithmic function that can be used as an alternative to piecewise defined step functions. It also shows how to use that function to create second tier logical functions such as multi-steps, boxcars, and valleys, and it provides a method to stitch the components of piecewise functions into a single equation.

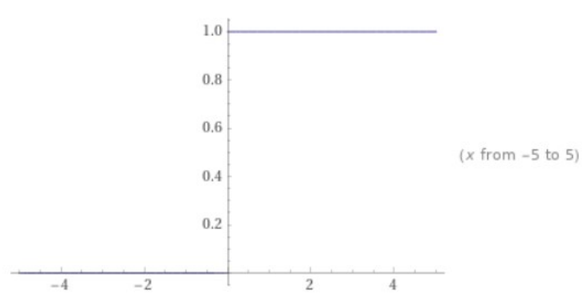
2 A Complex Logarithmic Step Function

While working with periodic functions to explore the prime distribution, I often found myself looking for a way to map certain numbers to either 0 or 1. This led to manipulating many functions, searching for ways to do so. During that time, I stumbled into expressing a unique equation that accomplishes this binary separation. Plotting different variants of equations, and tweaking parameters, I was a bit surprised when I found it. At a first glance of the equation, most wouldn't expect it to take the graphical form that it does, contributing to the reasons as to why I consider it unique. Not to mention, it's also relatively simple, basically invoking only pi, i, and the natural log; a curious form indeed.

2.1 The Basic Step Function

There isn't much more to be said about the history of the development of the function, and it serves at this point to simply state it outright, that the reader may become familiar with it, and discuss it further. The basic function is:

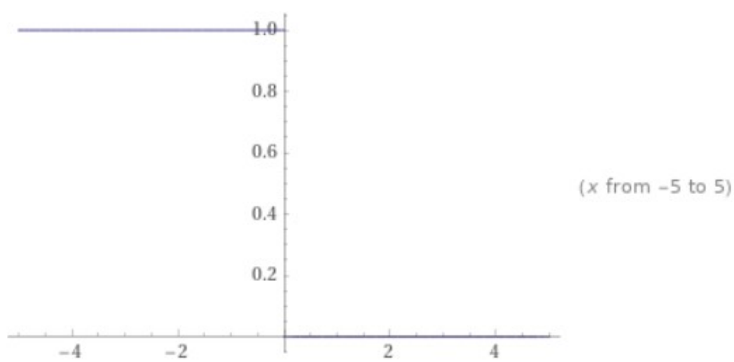
$$F(x) = \frac{\pi + (i * \ln(i * (-x))) - (i * \ln(i * (x)))}{2\pi} \tag{1}$$



As the graph shows, it takes the traditional step function shape, set to a height of 1, with the floor along the axis. There really isn't much more to it, until one looks closely at the form of the equation.

The first thing to notice, is the variable x inside the natural log. The log exists for all real values x, except when x = 0, in which case it triggers an undefined value. This is discussed in a subsection below. The next thing to notice, is that an i can be factored out, and the logs combined and simplified, giving ln(-1) = $i\pi$. With further simplification of the factored i and lone π in the numerator, F(x) simplifies to 0. What is being shown in the graph is technically the real portion of a complex output, and this becomes more clear when looking at the full form of the step function. For now, the last thing to note, is what happens when the i inside the log is removed, equation 2. The step flips over the y axis as shown.

$$F(x) = \frac{\pi + (i * \ln(-x)) - (i * \ln(x))}{2\pi} \tag{2}$$

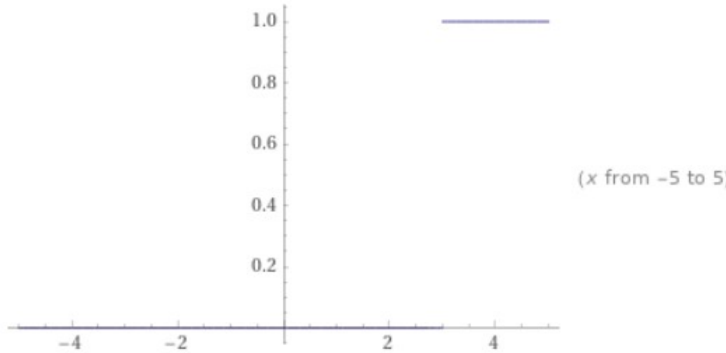


2.2 The Full Form of the Function

While the basic version can be scaled with a multiplier to make the step any height, one may want to move the step to a desired location. This is accomplished with the introduction of a parameter k into the log. The full function is equation 3, and the step in the example is moved to x = 3 by making k = 3. It is shown on the graph on the next page.

$$F(x) = \frac{\pi + (i * \ln(i * (k - x))) - (i * \ln(i * (-k + x)))}{2\pi} \tag{3}$$

Of course one can also scale the full function, and one could have added the parameters to the flipped version, making the step travel in the opposite direction. Another option for the flipped version, is to leave the i inside, and to simply exchange x for -x. This turns out to be a more consistent approach, due to the requirement of an i inside the log for many of the full constructions



to work. Otherwise, everything works as expected, and it's time to look at the limits, symbolic evaluation, and the case when the logs are evaluated at 0.

2.3 Symbolic Evaluation and the Limits

When centered at 0, the left and right hand limits of the function behave like a traditional Heaviside step. That is, the limit from the left is 0, and from the right it is 1. What about at 0? If the limit is handled explicitly, it returns an undefined, does not exist result as shown in the calculations below.

$$\lim_{x\to 0} \frac{\pi + i\log(i\,(0-x)) - i\log(i\,(0+x))}{2\,\pi}$$

Limit:

(two-sided limit does not exist)

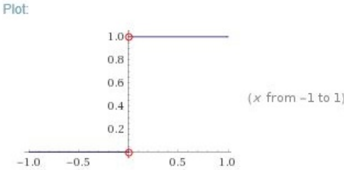
Limit from the left:

$$\lim_{x\to 0^-} \frac{\pi + i\log(i\,(0-x)) - i\log(i\,(0+x))}{2\,\pi} = 0$$

Limit from the right:

$$\lim_{x\to 0^+} \frac{\pi + i\log(i\,(0-x)) - i\log(i\,(0+x))}{2\,\pi} = 1$$

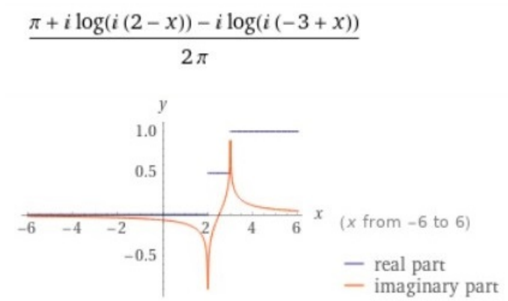
Plot:



Instead of running it explicitly, what happens when it's handled symbolically prior to calculation? When both logs are 0, the function becomes equation 4. Treating the log quantity as a symbolic whole, allows it to cancel with itself. That is, let ln(x) = q, then ln(0) - ln(0) = q - q = 0. This then makes F(x) equal to 1/2, which allows further calculations to continue, and it is conveniently equal to the half-maximum definition of the Heaviside version.

$$F(x) = \frac{\pi + i * (\ln(0) - \ln(0))}{2\pi} = \frac{\pi + i * 0}{2\pi} = \frac{1}{2} \tag{4}$$

Something else to consider is the use of 2 different constants for k instead of one, that is, letting the k in the first log be different from the one in the second log. This is mentioned here for 2 reasons. One is to help show how the step is coming from the real part, and 2 is to simply show and comment on the effect that it has. The graph below has the first k set to 2 and the second to 3.



Instead of a single step, there are now 2 steps, one at 2, and one at 3, with the stair in the middle having a value of 1/2. The imaginary portion now shows on the graph, and it's clear to see that the steps are coming from the real portion. This further supports the symbolic limit, as were the value of the first and second k to come together, the middle stair with value 1/2 shrinks to the point where the logs are symbolically evaluated at 0 to produce 1/2. This method also acts as an alternative to, or simplification of, the multi-step logical construction given in the next section. However, for consistency, the remainder of the constructions are all built with the full step, as it is easier to predict the effect of combining functions. As such, only a single, shared k is used and discussed throughout the paper.

To summarize, the 3 key takeaways to understanding why the function is presented in the form in which it is, are as follows. One, it allows the logs to be evaluated symbolically when their arguments are both 0. Two, the i is left inside the logs, and the logs are not simplified, so that real results are produced when taken in product with the i's outside of the logs; thus creating the steps. And three, a single parameter is used in order to guarantee a basic single step shape as the primary building block going forward.

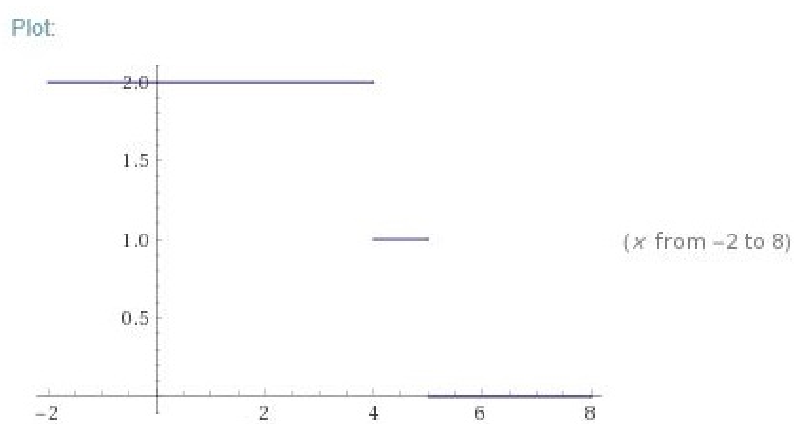
3 Logical Constructions

With the full step function, and an understanding of how it is handled symbolically, it is possible to create more complicated shapes. This section covers the 3 basic constructions that can be made with a step, a second order example made from the first 3, and it finishes with a brief comment on the function in 3-D. Each subsection is fairly straight forward, more formulaic in nature, and requires little commentary once the general idea of what is being presented is understood. The examples in each act to serve as a primary explanation. The first shape is a multi-step.

3.1 Multi-steps

Multi-steps are made from 2 or more single steps with offset transitions, such that the inter transition portion of the steps both interact constructively. In the example, equation 5, two left facing steps, one at x = 4, and one at x = 5, are added together.

$$\frac{\pi + (i * \ln(i * (-4 + x))) - (i * \ln(i * (4 - x)))}{2\pi} + \frac{\pi + (i * \ln(i * (-5 + x))) - (i * \ln(i * (5 - x)))}{2\pi} \tag{5}$$



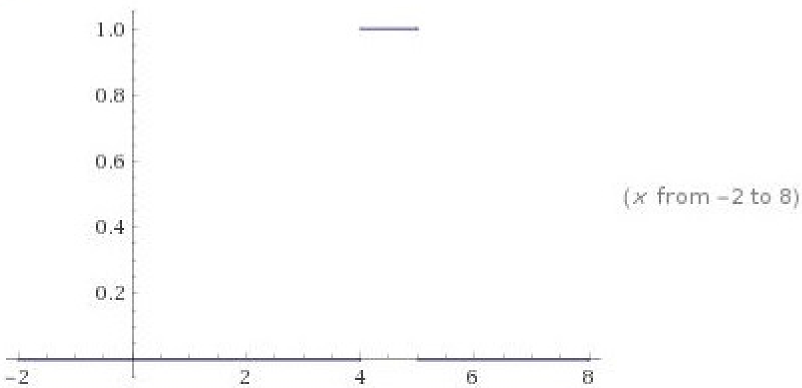
The result is that the portions to the left of x = 4, which are both equal to 1, combine to a value of 2, the portions to the right of x = 5, which are both 0, combine to remain 0, while the middle section, where only 1 of the steps takes value, combines to 1. This is similar to the effect gained and mentioned in the previous section of using different values for k.

3.2 Boxcar or Plateau

Boxcars, or plateaus, are made from 2 or more single steps with offset transitions, such that the inter transition portion interacts constructively, but the steps interact destructively. In the example, equation 6, a right facing step at x = 5 is subtracted from a right facing step at x = 4.

$$\begin{aligned} &\frac{\pi + (i * \ln(i * (4 - x))) - (i * \ln(i * (-4 + x)))}{2\pi} \\ -&\frac{\pi + (i * \ln(i * (5 - x))) - (i * \ln(i * (-5 + x)))}{2\pi} \end{aligned} \tag{6}$$

Plot:

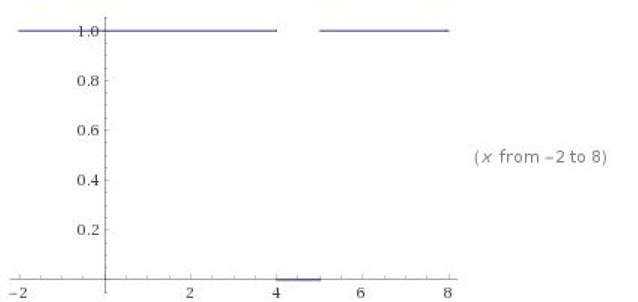


The result is that the portions to the left of x = 4, and to the right of x = 5, combine to make 0, while the middle section combines to 1.

3.3 Valleys

Valleys are the opposite of plateaus. They are made from 2 or more single steps with offset transitions, such that the inter transition portion interacts destructively, but the steps interact constructively. In the example, equation 7, a left facing step at x = 4 is added to a right facing step at x = 5.

$$\begin{aligned} &\frac{\pi + (i * \ln(i * (-4 + x))) - (i * \ln(i * (4 - x)))}{2\pi} \\ +&\frac{\pi + (i * \ln(i * (5 - x))) - (i * \ln(i * (-5 + x)))}{2\pi} \end{aligned} \tag{7}$$

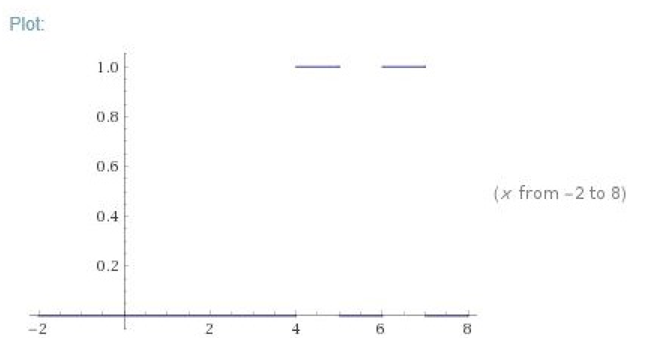


The result is that the portions to the left of x = 4, and to the right of x = 5, combine to make 1, while the middle section goes to 0. A quick note, boxcars and valleys with unit heights can quickly be turned into each other by running them through a linear filter which swaps 0s and 1s, such as y = -x + 1.

3.4 Further Combinations

Using the tools described so far, more specific second order shapes are formed. As an example, 4 steps are combined to create a single function with 2 boxcars and a valley in between.

$$\begin{aligned} &\frac{\pi + (i * \ln(i * (4 - x))) - (i * \ln(i * (x - 4)))}{2\pi} \\ &- \frac{\pi + (i * \ln(i * (5 - x))) + (i * \ln(i * (x - 5)))}{2\pi} \\ &+ \frac{\pi + (i * \ln(i * (6 - x))) - (i * \ln(i * (x - 6)))}{2\pi} \\ &- \frac{\pi + (i * \ln(i * (7 - x))) + (i * \ln(i * (-5 + x)))}{2\pi} \end{aligned} \tag{8}$$

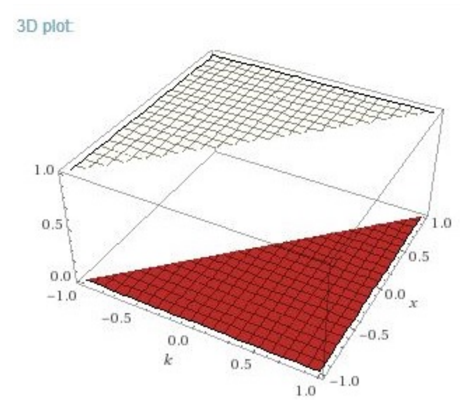


While not explored further here, summations over the full step using an index of k, while also using a function of k within the step, can quickly create periodic functions, power and exponentially spaced functions, and many other designs. Before proceeding, a brief comment is made about the step function as a surface in 3-D.

3.5 In 3-D

Instead of treating the function as y of x with parameter k, it can be thought of as the surface Z(k,x). This shows how the step follows x = k in 2-D, and creates its own step along that line in 3-D.

$$Z(k,x) = \frac{\pi + (i * \ln(i * (-k + x))) - (i * \ln(i * (k - x)))}{2\pi} \tag{9}$$



Of course the concept can be formally generalized to the 3rd or higher dimensions, however that is not approached here.

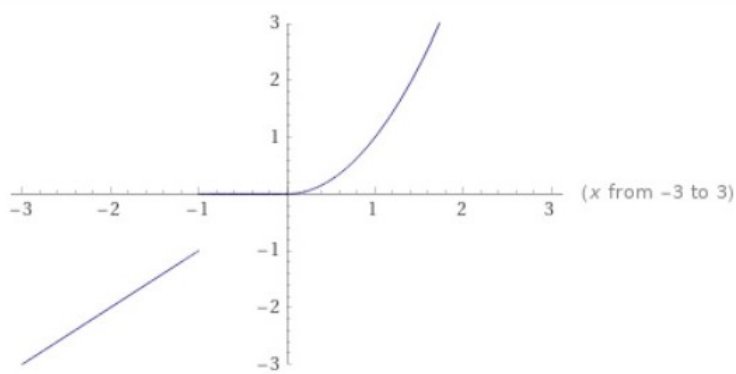
4 Piecewise Defined Function Stitching

These designs can now be used to stitch together the components of piecewise defined functions into a single function. Piecewise defined function stitching is the act of taking the component functions for each domain in a piecewise function, and combining them into a single function with a single domain. As an example, consider the following piecewise defined function.

$$f(x) = \begin{cases} x & x < -1 \\ 0 & -1 \le x \le 0 \\ x^2 & x > 0 \end{cases} \tag{10}$$

In order to stitch this together, one can use a left facing step at - 1, and a right facing step at 0. The left step is 1 when x < -1, and 0 to the right of that, so multiplying that step by the first function makes values < -1 take the value of the first function, and values to the right equal to 0. Since the right facing step is already 0 along the interval between -1 and 0, and this matches the middle function, this fact is utilized, and nothing more needs to be done for the second function. Then, since the right step is 0 when x < 0, and 1 to the right of that, multiply the right facing step by the 3rd function in the list, namely x^2. That makes the values left of 0 equal to 0, and those to the right equal to the function. Adding those two adapted steps together results in the following equation and graph.

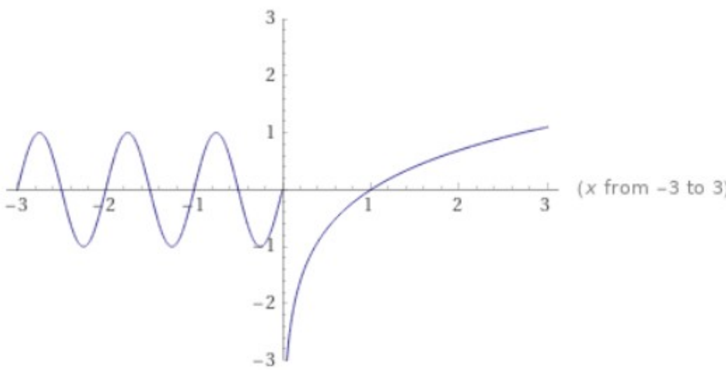
$$F(x) = \frac{x * (\pi + (i * \ln(i * (1 + x))) - (i * \ln(i * (-1 - x))))}{2\pi} + \frac{x^2 * (\pi + (i * \ln(i * (0 - x))) - (i * \ln(i * (-0 + x))))}{2\pi} \tag{11}$$



The single function now graphs all 3 of the piecewise components. The only consideration is of course the value at x = -1, which will take the average value of the functions it bridges when following symbolic evaluation, or in this case, take the value of the 2nd adapted step when the undefined 1st adapted step is chosen to be explicitly ignored.

In this manner, by combining the various constructs adjusted for specific domains, often only steps are needed, with their corresponding functional components, piecewise functions can be stitched into a single function. This technique lends itself to many piecewise functions with the main shortcoming being point-wise defined functions; that is functions which take specific values at many different single points within a domain. As one final simple visual example, here is sine stitched with the natural log at x = 0.

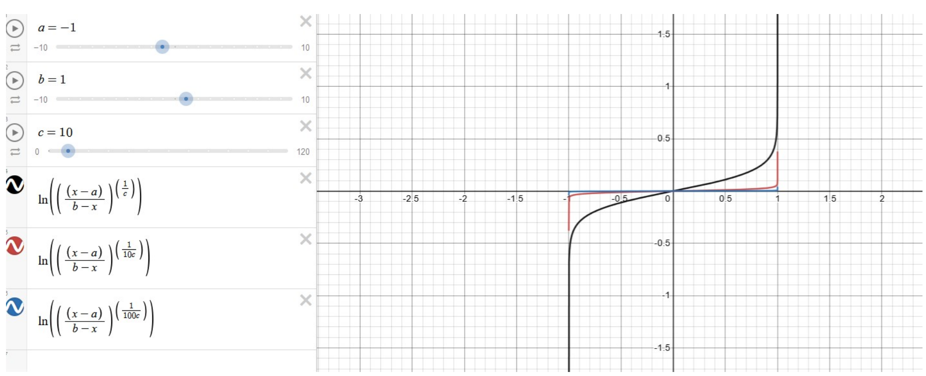
$$F(x) = \frac{\sin 2\pi x * (\pi + (i * \ln(i * x)) - (i * \ln(-i * x)))}{2\pi} + \frac{\ln x * (\pi + (i * \ln(-i * x))) - (i * \ln(i * x)))}{2\pi} \tag{12}$$



5 A Rapidly Converging Step Approximation

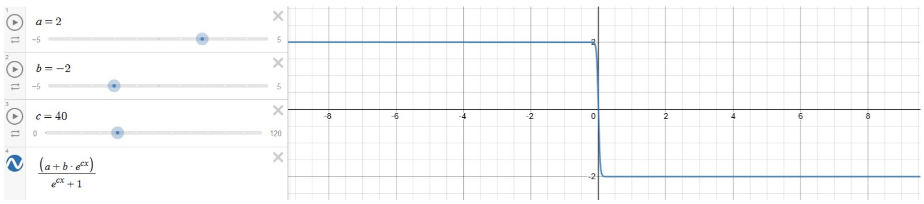
While working with the method, I stumbled upon one other noteworthy function. It's mentionable due to its related purpose, simple form, and rapid convergence. Consider G(x), equation 13.

$$G(x) = \ln((\frac{x-a}{b-x})^{\frac{1}{c}}) \tag{13}$$



G(x) has vertical asymptotes at a and b, set to -1 and 1 in the graph, making it very easy to set their locations and the distance between them. Then examine what happens by increasing the c parameter. With c = 10, the rate of change of the log is still visible, but by c = 100, it has almost disappeared. By c = 1,000, the cusps are relatively completely square at scale, and the rate of change from horizontal to vertical in that region is very rapid for such a simple function. This can be used as a sort of containment boxcar, and can be set to any height simply by adding a constant to lift the graph up the y axis. Perhaps more useful though, it can be rotated 90 degrees and used as a step function approximation, with accuracy and precision set by c. Swapping x and y, and then solving for y gives equation 14, with e as the exponential.

$$G(x) = \frac{a + be^{c(x-k)}}{e^{c(x-k)} + 1} \tag{14}$$



A and b still control the asymptotes, c the precision and accuracy, and equation 14 also has the parameter k added to it, so that the position along the x axis can easily be set. To get a sense of just how good of an approximation G(x) is, and how fast it converges, consider equation 14 with a = 2, b = -2, c = 100, and k = 0. At x = 1, the function is equal to 2 within 43 decimal places, and at x = 10, it's accurate to within 434 decimal places. It only gets better from there. In fact, adding a decimal place to c, has about 10 times the effect on the accuracy. A c of only 10,000, at x = 1, is already accurate to within 4,343 decimal places!

6 Conclusion

This concludes the general methods described in this paper for generating a unique complex logarithmic step function, using it to create other logistical constructions, and using it to stitch the components of piecewise defined functions into a single function. It also included and briefly discussed a rapidly converging step approximation. I hope you enjoyed the strategies, and if you found any errors within this technique, or would like to discuss it further, please contact me.