The Requirements on the Non-trivial Roots of the Riemann Zeta via the Dirichlet Eta Sum

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Abstract

An explanation of the Riemann Hypothesis is given in 8 parts, with the first being a statement of the problem. In the next 3 parts, the complex valued Dirichlet Eta sum, a known equivalence to Riemann Zeta in the critical strip, is split into 8 real valued sums and 2 constants. Part 5 explains a recursive relationship between the 8 sums. Section 6 shows that the sums must individually equal 0, and part 7 details conditions generated from the recursive system. Finally, part 8 solves the system in terms of the original inputs of the Dirichlet Eta sum. The result shows that the only possible solution for the real portion of the complex input, commonly labeled a, is that it must equal 1/2, and thus proves Riemann's suspicion.

1 A Statement of the Problem, and the General Approach to the Solution

The explanation begins with a well known version of the hypothesis based on the closely related Dirichlet Eta function. In that version, the Dirichlet Eta sum $\eta(s)$ is stated in a functional equation with the Riemann Zeta function, in order to analytically continue the domain of the Zeta function, and it is shown as equation 1.

$$\eta(s) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \left(1 - 2^{1-s}\right) \zeta(s) \tag{1}$$

Using the Dirichlet Eta sum, the Riemann hypothesis is often stated as "all the zeros of the Dirichlet eta function, falling in the critical strip $0 < \Re(s) < 1$, lie on the critical line $\Re(s) = 1/2$," where $\Re(s)$ is the real portion of the complex input s. That real portion is commonly labeled as lower case a.

So what is the nature of the zeros of the Eta function? The Eta function is an infinite sum of fractions, sometimes totaling to zero, where the denominator of that fraction sequence is the changing index of the sum raised to a complex valued power s. Small s is a standard complex number given as a + bi. The numerator of the sum's fraction also contains information. In this case, it's a negative 1 raised to a power involving the index, which causes the fraction to alternate between positive and negative. The goal then, and challenge of the problem, is to explain why the value of a, in the domain between 0 to 1, must be 1/2, and only 1/2, in order for that entire infinite sum of fractions to sum to zero. This is stated as equation 2.

$$\eta(s) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{a+bi}} = 0 \tag{2}$$

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The first major step is to separate the real and imaginary portions of the complex Eta sum, so that there is no longer a complex number inside the sum, but rather 2 real valued sums instead.

2 Separating the Real and Imaginary Portions of the Complex Sum

Start by using exponent rules on the index raised to a complex power, a + bi; equation 3.

$$n^s = n^{a+bi} = n^a n^{bi} \tag{3}$$

Then expand the complex exponent n^{bi} with Euler's well known formula. The result is shown in equation 4.

$$n^s = n^a \left(\cos\left(b \ln n\right) + i \sin\left(b \ln n\right)\right) \tag{4}$$

Put the now expanded form back into equation 2, and then express the numerator as a complex number, equation 5. Please also note, that I changed the n-1 to n+1 out of personal preference of convention, as I had used it while working the problem out on paper. This is allowed, as it does not change any of the values. That is, $(-1)^{n-1}$ will always equal $(-1)^{n+1}$ over the integer index.

$$\frac{(-1)^{n+1}}{n^s} = \frac{(-1)^{n+1} + 0i}{n^a \cos\left(b \ln n\right) + n^a \sin\left(b \ln n\right) i} \tag{5}$$

Next, use the general formula for dividing complex numbers, equation 6, to carry out the division shown in 7.

$$\frac{u+vi}{x+yi} = \frac{(ux+vy)+(vx-uy)\,i}{x^2+y^2} \tag{6}$$

$$\frac{(-1)^{n+1}+0i}{n^a\cos(b\ln n)+n^a\sin(b\ln n)\,i} =$$
$$\frac{(-1)^{n+1}n^a\cos(b\ln n)}{(n^a\cos(b\ln n))^2+(n^a\sin(b\ln n))^2} + \frac{0-(-1)^{n+1}n^a\sin(b\ln n)}{(n^a\cos(b\ln n))^2+(n^a\sin(b\ln n))^2}i \tag{7}$$

The result can be simplified by factoring out a n^a and by using trig rules on the sin squared plus cos squared in the denominator. The complex input Dirichlet Eta sum can now be expressed as the sum-difference of 2 sums with only real inputs, equation 8.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(b\ln n)}{n^a} - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(b\ln n)}{n^a}i = 0 \tag{8}$$

Notice that the left sum is real valued and deals with cosines, and that the right sum, though still sitting in front of the imaginary number i, is real valued in magnitude and deals with sines. Since the Dirichlet Eta sum is a sum of complex numbers, the result is also complex, which is expected. Therefore, in order for the original complex Eta sum to equal zero, and thus have a root, both the real and complex parts of its total must be zero. That is, equal to 0 + 0i.

After factoring out and dividing away a constant -1 from equation 8, the results are the 2 sums, equations 9 and 10, labeled A and B as follows.

$$A = \sum_{n=1}^{\infty}\frac{(-1)^n\cos(b\ln n)}{n^a} \tag{9}$$

A is referred to as the real portion of the complex Dirichlet Eta sum.

$$B = \sum_{n=1}^{\infty}\frac{(-1)^n\sin(b\ln n)}{n^a} \tag{10}$$

B is referred to as the imaginary portion of the complex Dirichlet Eta sum, though its magnitude is real valued.

Now, the task becomes to determine when these 2 new sums are both zero at the same time. To do that, they will need to be broken down, and the first stage for such, is separating each of them into their even and odd parts.

3 Separating the Even and Odd Portions of Both the Real and Imaginary Sums

Instead of using one sum for each of A and B, as they are stated thus far, and instead of letting their indices n run over the full set of integers, use 2 sums for each, separating the even and odd inputs of the indices. Do this by separating n into 2n, for the evens, and into 2n-1, for the odds. This is shown for both A and B in equations 11 and 12.

$$A = \sum_{n=1}^{\infty} \frac{(-1)^n \cos(b \ln n)}{n^a} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} \cos(b \ln(2n-1))}{(2n-1)^a} + \sum_{n=1}^{\infty} \frac{(-1)^{2n} \cos(b \ln 2n)}{(2n)^a} = 0 \quad (11)$$

$$B = \sum_{n=1}^{\infty} \frac{(-1)^n \sin(b \ln n)}{n^a} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} \sin(b \ln(2n-1))}{(2n-1)^a} + \sum_{n=1}^{\infty} \frac{(-1)^{2n} \sin(b \ln 2n)}{(2n)^a} = 0 \quad (12)$$

The behavior of -1 raised to even or odd powers allows the resulting sums of equations 11 and 12 to be simplified, obtaining 13 and 14 respectively.

$$A = \sum_{n=1}^{\infty} \frac{\cos(b \ln 2n)}{(2n)^a} - \sum_{n=1}^{\infty} \frac{\cos(b \ln(2n-1))}{(2n-1)^a} = 0 \quad (13)$$

$$B = \sum_{n=1}^{\infty} \frac{\sin(b \ln 2n)}{(2n)^a} - \sum_{n=1}^{\infty} \frac{\sin(b \ln(2n-1))}{(2n-1)^a} = 0 \quad (14)$$

The sums involving 2n are known as the even portions, and the sums with 2n-1, the odd portions. Notice that in both cases it is the even sums minus the odd sums. Specifically labeling the 4 sums from equations 13 and 14 gives equations 15 through 18.

$$A_{even} = A_e = \sum_{n=1}^{\infty} \frac{\cos(b \ln 2n)}{(2n)^a} \quad (15)$$

A_e is referred to as the real even portion.

$$A_{odd} = A_o = \sum_{n=1}^{\infty} \frac{\cos(b \ln(2n-1))}{(2n-1)^a} \quad (16)$$

A_o is referred to as the real odd portion.

$$B_{even} = B_e = \sum_{n=1}^{\infty} \frac{\sin\left(b \ln 2n\right)}{\left(2n\right)^a} \tag{17}$$

B_e is referred to as the imaginary even portion.

$$B_{odd} = B_o = \sum_{n=1}^{\infty} \frac{\sin\left(b \ln\left(2n-1\right)\right)}{(2n-1)^a} \tag{18}$$

B_o is referred to as the imaginary odd portion.

This isn't yet broken down enough, and in order to determine when these new sum-differences in the real and imaginary sums are equal to zero, they must be deconstructed further. However, the odd sums do not lend themselves to being broken down easily, if possibly at all. Luckily, the even sums do, and later, functional relationships for the odd sums will be found so that they can be handled. In the mean time, the next main phase of the explanation requires separating the Sine and Cosine portions of the even parts.

4 Separating the Sin and Cos Portions of the Real Even and Imaginary Even Sums

To separate the even sums, begin with the ln(2n) using log rules, equation 19, and follow up with the trigonometry formulas for addition within Cosines and Sines, equations 20 and 21. The initial results are then shown in 22 and 23.

$$\ln 2n = \ln 2 + \ln n \tag{19}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{20}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \tag{21}$$

$$A_e = \sum_{n=1}^{\infty} \frac{\cos(b \ln 2)\cos(b \ln n) - \sin(b \ln 2)\sin(b \ln n)}{2^a n^a} \tag{22}$$

$$B_e = \sum_{n=1}^{\infty} \frac{\sin(b \ln 2)\cos(b \ln n) + \cos(b \ln 2)\sin(b \ln n)}{2^a n^a} \tag{23}$$

In this case, 22 and 23 have addition and subtraction over a common denominator, so they can each be separated into yet another 2 sums. Those resulting sums take the form of products of functions of a and b independent of the index, multiplied by a portion of the sum dependent on the index, and therefore, those independent portions that include a and b can be pulled out in front as constants. This is shown as equations 24 and 25.

$$A_e = \left(\frac{\cos(b\ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\cos(b\ln n)}{n^a}\right) - \left(\frac{\sin(b\ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\sin(b\ln n)}{n^a}\right) \tag{24}$$

$$B_e = \left(\frac{\sin(b\ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\cos(b\ln n)}{n^a}\right) + \left(\frac{\cos(b\ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\sin(b\ln n)}{n^a}\right) \tag{25}$$

Next, respectively label K_c and K_s for the new cosine based and sine based constants in equations 24 and 25, shown as 26 and 27.

$$K_c = \frac{\cos(b\ln 2)}{2^a} \tag{26}$$

$$K_s = \frac{\sin(b\ln 2)}{2^a} \tag{27}$$

Also, label the 2 different sums amongst 24 and 25, noting that the K constants are the same for the real even, A_e, and imaginary even, B_e, sums, only in different positions. These are equations 28 and 29.

$$C = \sum_{n=1}^{\infty} \frac{\cos(b\ln n)}{n^a} \tag{28}$$

This is known as the basic cosine sum.

$$S = \sum_{n=1}^{\infty} \frac{\sin(b\ln n)}{n^a} \tag{29}$$

This is known as the basic sine sum.

Now finally, between the 10 terms A, B, A_e, A_o, B_e, B_o, K_c, K_s, C, and S, there is enough information to determine when the original infinite complex valued Dirichlet Eta sum is equal to zero, and to answer why the real variable a must be 1/2. In order to do that, the next step is to understand what maintains an output of 0 throughout splitting the original Dirichlet Eta sum into 8 other sums and 2 constants.

5 The Recursive Functional Relationships Between the Sums

For the remaining sections, the indices and upper bounds of the sums do not change, and have mostly been omitted for brevity and visual clarity, as they do not affect the relationships or outcomes.

The self referential relationship amongst the sums is generally stated in words as follows. The real and imaginary sums are broken into even and odd sums, then, the even sums are broken into sine and cosine sums. However, those new sine and cosine sums end up being composed in terms of the earlier even and odd parent sums, and thus include a loop.

Stating the relation from equation 13, using 15 and 16, gives equation 30, which is the even and odd split of the real portion.

$$A = A_e - A_o \tag{30}$$

Likewise, stating the relation from 14, using 17 and 18, gives 31, which is the even and odd split of the imaginary portion.

$$B = B_e - B_o \tag{31}$$

With equation 8, it was noted that the sums A, eq.9, and B, eq.10, must both be 0, and this is stated again with eq.30 and eq.31 as requirements in the equations in 32.

$$A = A_e - A_o = 0 \quad \text{AND} \quad B = B_e - B_o = 0 \tag{32}$$

This leads to the requirement in 33.

$$A_e = A_o \quad \text{AND} \quad B_e = B_o \tag{33}$$

Using the labels from equations 26-29, 24 and 25 are written as 34 and 35.

$$A_e = K_c C - K_s S \tag{34}$$

$$B_e = K_s C + K_c S \tag{35}$$

Now, review and more closely examine equations 28 and 29. Do the cosine and sine sums look familiar? They sure look like the real sum A, eq.9, and the imaginary sum B, eq.10, except for the -1 raised to the power, that is, except for the alternating part. In fact though, that is exactly what they are! The alternating real and imaginary sums, eqs.9 and 10, subtract out every other term, while the sine and cosine sums, eqs. 28 and 29, add all the terms, of an otherwise identical sum. What are those other terms, which are being subtracted in the case of the real and imaginary sums, but are being added in the case of the sine and cosine sums? Equations 13 and 14 show that those terms turn out to be the odd function sums! That is, the real and imaginary sums are the difference of their respective even and odd sums, while the cosine and sine sums are the sum of their respective even and odd sums. This gives equations 36 and 37.

$$C = A_e + A_o \tag{36}$$

$$S = B_e + B_o \tag{37}$$

Adding 2 copies of the corresponding odd function to each side of the equations in 32, using A and B in terms of the even and odd sums, and then substituting with equations 36 and 37 respectively, gives the 2 sets of equations shown in 38. This is the same as using eq.33 with eqs.36 and 37.

$$C = A_e + A_o = 2A_o \quad \text{AND} \quad S = B_e + B_o = 2B_o \tag{38}$$

Eq.33 requires $A_e = A_o$ and $B_e = B_o$, so that it can also be written as eq.39.

$$C = 2A_e = 2A_o \quad \text{AND} \quad S = 2B_e = 2B_o \tag{39}$$

From this information, equation 33 can be split into 2 cases. At a minimum, eq.33 shows that corresponding even and odd sums must have the same value. Let case one be a shared value of 0, and let case 2 be sharing any value other than 0. The next section shows that it must be case one, and that all the sums must individually be 0.

6 Showing that the Sums Must Individually be Zero

Using eq.39, and substituting into 34 and 35, gives 40 and 41.

$$C = 2(K_c C - K_s S) \tag{40}$$

$$S = 2(K_s C + K_c S) \tag{41}$$

Since the K values are constants to the sums, this can now be treated as a system of 2 equations and 2 unknowns. Solving for S in eq.40 gives the following.

$$S = \frac{\left(K_c - \frac{1}{2}\right)}{K_s} C \tag{42}$$

Substituting 42 into 41 to solve the system, and simplifying, leaves 43.

$$\left(K_c^2 - K_c + K_s^2 + \frac{1}{4}\right) C = 0 \tag{43}$$

This shows that either C is 0, the portion in the parentheses is 0, or both parts are 0. In either case where C is 0, it means from eq.39 and eq.42 that case 1 must be true, and therefore that all sums must be 0. Using the quadratic equation on the portion within the parentheses for K_c gives eq.44.

$$K_c = \frac{1 \pm \sqrt{-4K_s^2}}{2} \tag{44}$$

From eq.27 it is known that K_s is real valued, and therefore its square will be positive. Similarly, from eq.26, K_c is real valued. Because of the -4 inside the square root, the only possible solution is for $K_s = 0$, which then makes $K_c = 1/2$. This creates a contradiction as follows.

If $K_s = 0$, eq.27 requires that $b \ln 2 = n\pi$, a multiple of pi, for some integer n. However, if $b \ln 2 = n\pi$, then the numerator of eq.26 is plus or minus 1, and from eq.44, you get the following.

$$K_c = \frac{\pm 1}{2^a} = \frac{1}{2} \tag{45}$$

This would then require that a is either complex valued or 1, which places it outside the domain of a, and therefore that the solutions within the domain of a, of which we know there are at least solutions when a=1/2, occur in case one when C is equal to 0. Therefore, all sums must individually equal 0, stated as 46.

$$A = B = A_e = A_o = B_e = B_o = C = S = 0 \tag{46}$$

Now that it's been determined that all 8 sums must be 0 in order to make the Dirichlet Eta 0, this allows systems of equations to be formed in terms of the sums. Those systems allow the conditions on K_c and K_s to be examined such that they can be solved for the requirements on a.

7 The Conditions Generated by the Systems of Sums

Using the fact that the sums are each 0, along with the fact that if 2 values are both 0 then the sum and difference of those 2 values are also both 0, 2 new systems of equations are created to examine what each says about K_c and K_s. Before doing this, first note that with sums equal to 0, eqs.34 and 35 suggest that the values of K_c and K_s are independent from the sums, and that they can be any value. That is, you can plug in whatever you like for the constants, and the equations will still hold true due to the 0 values for A_e, B_e, C, and S. This is the same for eq.43. It will be shown for both systems, as it was in eq.43, that there are no solutions for a within its domain, and that individually, either system would seem to indicate that K_c and K_s are independent of the sums and free to take any value. In fact they are, and it is the opposite that is true, rather that the sums are somehow dependent on those constants as seen in eqs.34 and 35. This raises the somewhat awkward question that even though it is known that the sums must be 0, and that the constants are independent of the sums, what restriction on the values of the constants insure that the even sums still equal 0, specifically and independently of whatever values the C and S functions take, all the while even though it is already known that C and S will take values of 0?

Another way to summarize or understand this question is to say that even though we have determined that the sums must equal 0, the constants "don't know that," and they must still take some value such as is appropriate to make the sums within the recursive relationships all settle to 0 at the same time. Yet a third way to view the question is to recognize that the constants are actually the 2nd terms of the infinite C and S sums, and to wonder what values the second terms must take in order to be the negative of the sums of all other terms, such that the overall sums then equal 0.

As it was stated above, neither system can do this alone. However, between the 2 systems, exists a relation independent of the sums that places requirements on the constants. This is exactly what is needed. Using the sum and difference of the Real and Imaginary sums, along with 46 and 32, gives the following requirement.

$$A+B=A_e-A_o+B_e-B_o=0 \quad \text{AND} \quad A-B=A_e-A_o-B_e+B_o=0 \tag{47}$$

Substitute in using 34 and 35 for the even sums, and eq.39 for the corresponding odd sums, to get 48 and 49.

$$K_cC-K_sS-\frac{1}{2}C+K_sC+K_cS-\frac{1}{2}S=0 \tag{48}$$

$$K_cC-K_sS-\frac{1}{2}C-K_sC-K_cS+\frac{1}{2}S=0 \tag{49}$$

Solving for C in 48 gives 50.

$$C=\frac{(-K_c+K_s+\frac{1}{2})}{(K_c+K_s-\frac{1}{2})}S \tag{50}$$

Plugging 50 into 49, and then simplifying, gives 51.

$$(4K_c^2-4K_c+4K_s^2+1)S=0 \tag{51}$$

Put that equation aside for the moment, and then repeat the process using the sum and difference of odd sums. Using eq.38 you get the following.

$$A_o+B_o=C-A_e+S-B_e=0 \quad \text{AND} \quad A_o-B_o=C-A_e-S+B_e=0 \tag{52}$$

Substitute in 52, again using 34 and 35 for the even sums.

$$C-K_cC+K_sS+S-K_sC-K_cS=0 \tag{53}$$

$$C-K_cC+K_sS-S+K_sC+K_cS=0 \tag{54}$$

Solving for C in 53 gives 55.

$$C=\frac{(K_c-K_s-1)}{(-K_c-K_s+1)}S \tag{55}$$

Plugging 55 into 54, and then simplifying, gives 56.

$$(2K_c^2 - 4K_c + 2K_s^2 + 2)S = 0 \tag{56}$$

Now examine eqs.51 and 56, and notice they both take the form of eq.43 except in terms of S instead of C. In fact, the quadratic in 51 is a scaled version of 43, has equivalent roots to the one in 43, and it means that the condition generated by using the sum and difference of the real and imaginary sums is the same as stating that the even sums are 0, as was the case in eq.43. From that, it is already known that there are no solutions within the domain, or specifically, that K_c=1/2, K_s=0, and a=1. What about eq.56?

Solving the quadratic in 56, and using the same logic as was employed in eq.43, gives the only solution of K_c=1, K_s=0, and a=0, which again places a outside of the original domain. Interestingly, we see a taking the values of the lower and upper limits just outside the closed domain, 0 and 1, and furthermore, that the average of those values is of course 1/2.

As the last step, it is now possible to solve the system independently of the sums, and to do the substitutions back into K_c and K_s, showing the requirements on a from the original Dirichlet Eta sum.

8 Solving the Systems Independently of the Sums

At this point, setting eq.51 equal to eq.56 allows the sum to be divided out as an unknown variable, and establishes a relation between the constants independent of any sum.

$$4K_c^2 - 4K_c + 4K_s^2 + 1 = 2K_c^2 - 4K_c + 2K_s^2 + 2 \tag{57}$$

Simplifying 57 gives 58.

$$K_c^2 + K_s^2 = \frac{1}{2} \tag{58}$$

Substitute back in for K_c and K_s using eqs. 26 and 27 to get 59.

$$\frac{\cos^2(b\ln 2)}{2^{2a}} + \frac{\sin^2(b\ln 2)}{2^{2a}} = \frac{1}{2} \tag{59}$$

Finally, there it is. Simplifying eq.59 using the familiar trig. identity leaves eq.60, leading to $2a = 1$, and therefore eq.61, a=1/2.

$$\frac{1}{2^{2a}} = \frac{1}{2} \tag{60}$$

$$a = \frac{1}{2} \tag{61}$$

This shows that there is indeed only one possible choice for a that allows the recursive system of sums and coefficients to balance independently of the sums, such that all 8 sums, and thus the original Dirichlet Eta function, are equal to 0.

Therefore, a must $= 1/2$, and Riemann's suspicions were correct!

Q.E.D.